

THE PSEUDO ORBIT TRACING PROPERTY AND EXPANSIVENESS ON UNIFORM SPACES

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ABSTRACT. Uniform space is a generalization of metric space. The main purpose of this paper is to extend several results contained in [5, 6] which have for an expansive homeomorphism with the pseudo orbit tracing property (POTP in short) on a compact metric space (X, d) for an expansive homeomorphism with the POTP on a compact uniform space (X, \mathcal{U}) .

we characterize stable and unstable sets, sink and source and saddle, recurrent points for an expansive homeomorphism which has the POTP on a compact uniform space (X, \mathcal{U}) .

1. Introduction and preliminaries

In 1987, Jerzy Ombach had characterized results concerning the class of expansive homeomorphisms having the pseudo orbit tracing property on a compact metric space (X, d) . Uniform space (X, \mathcal{U}) is a generalization of metric space (X, d) .

The main purpose of this paper is to extend some main theorems contained in [5] for an expansive homeomorphism with the POTP on a compact uniform space (X, \mathcal{U}) .

We prove here Theorem 2.3, Theorem 3.10 and Theorem 4.1 for an expansive homeomorphism $f : (X, \mathcal{U}) \rightarrow (X, \mathcal{U})$ which has the POTP on a compact uniform space (X, \mathcal{U}) .

We now introduce a definition of uniform space.

DEFINITION 1.1. A uniform structure or uniformity \mathcal{U} on a set X is a collection of subsets of $X \times X$ satisfying the following properties:

- (1) Each member of \mathcal{U} contains the diagonal Δ .
- (2) If $\alpha \in \mathcal{U}$ and $\alpha \subset \beta \subset X \times X$, then $\beta \in \mathcal{U}$.

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- (3) If α and β are members of \mathcal{U} , then $\alpha \cap \beta$ is also a member of \mathcal{U} .
 (4) If $\alpha \in \mathcal{U}$, then $\alpha^{-1} = \{(y, x) \mid (x, y) \in \alpha\} \in \mathcal{U}$.
 (5) For any $\alpha \in \mathcal{U}$, there is $\beta \in \mathcal{U}$ such that $\beta^2 = \beta \circ \beta \subset \alpha$, where

$$\beta \circ \beta = \{(x, y) \mid \text{there exists } z \in X \text{ with } (x, z) \in \beta, (z, y) \in \beta\}.$$

The set X equipped with a uniformity \mathcal{U} is called a uniform space and an element of \mathcal{U} called an entourage of X . An entourage α of X is called symmetric if $\alpha^{-1} = \alpha$.

REMARK 1.2. If X is a uniform space, then the topology \mathfrak{S} on X induced by \mathcal{U} is the collection of all subsets U of X such that for each $x \in U$ there is $\alpha \in \mathcal{U}$ with $\alpha[x] = \{y \mid (x, y) \in \alpha\} \subset U$, and a uniform space X is Hausdorff if and only if $\bigcap_{\alpha \in \mathcal{U}} \alpha \in \Delta_X$.

More details for uniform spaces is contained in [4].

DEFINITION 1.3. Let X be a uniform space and $f : X \rightarrow X$ be a homeomorphism. Let α and β be entourages of X . A sequence of points $\{x_n\}_{n \in \mathbb{Z}}$ is called a β -pseudo orbit for f if $(f(x_n), x_{n+1}) \in \beta$ for all in $n \in \mathbb{Z}$. A sequence $\{x_n\}_{n \in \mathbb{Z}}$ is α -traced if there is a point $x \in X$ such that $(f^n(x), x_n) \in \alpha$ for all in $n \in \mathbb{Z}$.

The homeomorphism f has the *pseudo orbit tracing property*(POTP in short) if for any entourage α there is an entourage β such that any β -pseudo orbit is α -traced. A homeomorphism f is said to be *expansive*(EXPS in short) if there is an entourage α_f such that $(f^n(x), f^n(y)) \in \alpha_f$ for all $n \in \mathbb{Z}$ and $x, y \in X$ implies $x = y$.

2. Stable and unstable sets

To prove main theorems we need the following Lemma 2.1 and Lemma 2.2, which we give without proofs.

LEMMA 2.1. *Let $f : X \rightarrow X$ be a homeomorphism on uniform space X and let $n \in \mathbb{Z} - \{0\}$. Then $f : X \rightarrow X$ has POTP if and only if $f^n : X \rightarrow X$ has POTP.*

Also $f : X \rightarrow X$ is EXPS if and only if $f^n : X \rightarrow X$ is EXPS.

Let $f : X \rightarrow X$ be a homeomorphism on uniform space X . For $N > 0$, we denoted by

$$V_N = \{(x, y) \in X^2 \mid (f^n(x), f^n(y)) \in \alpha \text{ for all } |n| \leq N \text{ and entourage } \alpha\}$$

LEMMA 2.2. *Let a homeomorphism $f : X \rightarrow X$ be expansive on uniform space X . Then*

- (1) *for every entourage α , there is $N \in \mathbb{N}$ such that $V_N \subset \alpha$,*
- (2) *for every $N \in \mathbb{N}$, there is entourage α such that $\alpha \subset V_N$.*

Let (x_n) and (y_n) be sequences on uniform space X .

If for any entourage α , there exists $n \in \mathbb{N}$ such that $(x_n, y_n) \in \alpha$, then we denote $(x_n, y_n) \rightarrow 0$ as $n \rightarrow \infty$.

Let $f : X \rightarrow X$ be a homeomorphism on uniform space X . Let $x \in X$ and an entourage α .

We define the local stable set and local unstable set of x by

- (a) $W_\alpha^s = \{y \in X \mid (f^n(x), f^n(y)) \in \alpha \text{ for all } n \geq 0\}$,
- (b) $W_\alpha^u = \{y \in X \mid (f^n(x), f^n(y)) \in \alpha \text{ for all } n \leq 0\}$.

And also, for $x \in X$ the stable set and unstable set are defined by

- (c) $W^s = \{y \in X \mid (f^n(x), f^n(y)) \rightarrow 0 \text{ for all } n \rightarrow \infty\}$,
- (d) $W^u = \{y \in X \mid (f^n(x), f^n(y)) \rightarrow 0 \text{ for all } n \rightarrow -\infty\}$.

THEOREM 2.3. *Let X be a compact uniform space and $f : X \rightarrow X$ be a homeomorphism. Assume that f is an expansive with the pseudo orbit tracing property.*

Then there is entourage α_0 such that for every entourage $\alpha \subset \alpha_0$, there is entourage β and continuous function $h : \beta \rightarrow X$ such that

- (A) *for every $x, y \in X$, $W_\alpha^s(x) \cap W_\alpha^u(y)$ consists at most one point ; for $(x, y) \in \beta$, $W_\alpha^s(x) \cap W_\alpha^u(y) = \{h(x, y)\}$,*
- (B) *for every $x \in X$,*
 $W_\alpha^s(x) \cap \beta[x] = \{y \mid y = h(x, y), (x, y) \in \beta\}$,
 $W_\alpha^u(x) \cap \beta[x] = \{y \mid y = h(y, x), (x, y) \in \beta\}$,
- (C) $W_\alpha^s(x) \subset W^s(x), W_\alpha^u(x) \subset W^u$,
- (D) $W^s(x) = \cup_{n=0}^\infty f^{-n}(W_\alpha^s(f^n(x)))$,
 $W^u(x) = \cup_{n=0}^\infty f^n(W_\alpha^u(f^{-n}(x)))$.

Proof. Let α_f be entourage for an expansive homeomorphism f . There exists entourage α_0 such that $\alpha_0^3 \subset \alpha_f$.

Let $\alpha \subset \alpha_0$. By the uniform continuity of f , there exists entourage $\beta_0 \subset \alpha$ such that $(x, y) \in \beta_0$ implies $(f(x), f(y)) \in \alpha$.

Since $f \in \text{POTP}$, there is entourage β such that every β -pseudo orbit is β_0 -traced. Fix $(x, y) \in \beta$.

Consider a sequence $\{x_n\}$ defined as

$$x_n = \begin{cases} f^n(y) & \text{for } n < 0 \\ f^n(x) & \text{for } n \geq 0, \end{cases}$$

which is a β -pseudo orbit.

By the POTP of f , it is β_0 -traced by some point $p \in X$.

We claim that there is only one such point. Suppose that p and p' are two such points. From $(f^n(p), x_n), (f^n(p'), x_n) \in \beta$ for all $n \in \mathbb{Z}$, we have $(f^n(p), f^n(p')) \in \beta_0^2 \subset \alpha^2 \subset \alpha_0^2 \subset \alpha_0^3 \subset \alpha_f$ for all $n \in \mathbb{Z}$. Thus we get $p = p'$.

Next, define a map $h : \beta \rightarrow X$ by $h(x, y) = p$ for each $(x, y) \in \beta$.

For each $(x, y) \in \beta$, we obtain the following

$$(f^n(x), f^n(h(x, y))) \in \beta_0 \subset \alpha \text{ for all } n \geq 0,$$

$$(f^n(y), f^n(h(x, y))) \in \beta_0 \subset \alpha \text{ for all } n < 0.$$

In particular, $(f^{-1}(y), f^{-1}(h(x, y))) \in \beta_0$. By the choice of β_0 , we get $(y, h(x, y)) \in \alpha$. This means that $(f(f^{-1}(y)), f(f^{-1}(h(x, y)))) = (y, h(x, y)) \in \alpha$. Hence $(f^n(y), f^n(h(x, y))) \in \alpha$ for all $n \leq 0$, so that $h(x, y) \in W_\alpha^s(x) \cap W_\alpha^u(y)$.

We claim that $h(x, y) = p$ is unique in $W_\alpha^s(x) \cap W_\alpha^u(y)$.

Assume that $p, p' \in W_\alpha^s(x) \cap W_\alpha^u(y)$ with $h(x, y) = p, h(x, y) = p'$. Then for every $n \geq 0$, by $(f^n(p), f^n(x)) \in \alpha^{-1} = \alpha$ and $(f^n(x), f^n(p')) \in \alpha$, we get $(f^n(p), f^n(p')) \in \alpha^2 \subset \alpha_0^2 \subset \alpha_0^3 \subset \alpha_f$. Also for every $n \leq 0$, by $(f^n(p), f^n(y)) \in \alpha^{-1} = \alpha$ and $(f^n(y), f^n(p')) \in \alpha$, we have $(f^n(p), f^n(p')) \in \alpha^2 \subset \alpha_0^2 \subset \alpha_0^3 \subset \alpha_f$. This means that $p = p'$ by expansiveness.

To prove the continuity of the mapping h we use Lemma 2.1. For each neighborhood U of $h(x, y)$, there exists entourage γ such that $\gamma[h(x, y)] \subset U$. By Lemma 2.2, we get $V_N \subset \gamma$ for some $N \in \mathbb{N}$. Since f^{-N}, \dots, f^N are uniformly continuous, there is entourage δ such that $(U, V) \in \delta$ implies $(f^k(U), f^k(V)) \in \gamma$ for all $|k| \leq N$.

Let $(x', y') \in \delta[x] \times \delta[y]$. From $(x, x') \in \delta$ and $(y, y') \in \delta$, we obtain $(f^n(x), f^n(x')) \in \gamma$ for all $0 \leq n \leq N$ and $(f^n(y), f^n(y')) \in \gamma$ for all $-N \leq n \leq 0$.

By $(f^n(h(x, y)), f^n(x)) \in \alpha^{-1} = \alpha$ and $(f^n(x'), f^n(h(x', y'))) \in \alpha$ for all $n \geq 0$, it follows that $(f^n(h(x, y)), f^n(h(x', y'))) \in \alpha^3 \subset \alpha_0^3 \subset \alpha$ for all $0 \leq n \leq N$.

Also by $(f^n(h(x, y)), f^n(y)) \in \alpha^{-1} = \alpha$ and $(f^n(y'), f^n(h(x', y'))) \in \alpha$ for all $n \geq 0$, it follows that $(f^n(h(x, y)), f^n(h(x', y'))) \in \alpha^3 \subset \alpha_0^3 \subset \alpha$ for all $-N \leq n \leq 0$.

Thus $(f^n(h(x, y)), f^n(h(x', y'))) \in \alpha_f$ for all $|n| \leq N$, so that

$(h(x, y), h(x', y')) \in V_N \subset \gamma$. This means that $h(x', y') \in \gamma[h(x, y)] \subset U$, i.e., h is continuous. Therefore (A) is proved.

To show (B), let us note first $y = h(x, y) \in W_\alpha^s(x)$ and for $y \in W_\alpha^s(x)$, we know $y \in W_\alpha^s(x) \cap W_\alpha^u(y)$. From $(x, y) \in \beta$, it follows that $y = h(x, y)$. The proof of the remaining case is similar.

Next we prove (C). Let $y \in W_\alpha^s(x)$. For each entourage γ , there exists $N \in \mathbb{N}$ such that $V_N \subset \gamma$ by Lemma 2.1. Put $n \geq N$. From $k + n \geq 0$ for all $|k| \leq N$, it follows that $(f^k(f^n(x)), f^k(f^n(y))) = (f^{k+n}(x), f^{k+n}(y)) \in \alpha \subset \alpha_f$ for all $|k| \leq N$. Thus $(f^n(x), f^n(y)) \in V_N \subset \gamma$ for all $n \geq N$, so that $(f^n(x), f^n(y)) \rightarrow 0$ as $n \rightarrow \infty$. Consequently, we conclude that $y \in W^s(x)$. By similar method, we get $W_\alpha^u(x) \subset W^u(x)$.

Finally, the proof of (D) is trivial. □

3. Sink, source and saddle

In this section we characterize three types (sink, source and saddle) of behaviour of orbits near a point on a compact uniform space (X, \mathcal{U})

We assume in the sequel that an expansive homeomorphism $f : X \rightarrow X$ which has the POTP on a compact uniform space $X = (X, \mathcal{U})$ and $\alpha \subset \alpha_0$, where α_0 is given satisfying the Theorem 2.3.

Define maps L^+, L^- from X into 2^X by setting for each $x \in X$,

$$L^+(x) = \{y \mid f^{n_k}(x) \rightarrow y \text{ for some } n_k \rightarrow \infty\},$$

$$L^-(x) = \{y \mid f^{n_k}(x) \rightarrow y \text{ for some } n_k \rightarrow -\infty\}.$$

For each $x \in X$, the set $L^+(x)$ is called its positive limit set, and the set $L^-(x)$ is called its negative limit set.

PROPOSITION 3.1. *If $\text{int}W^s(x) \neq \emptyset$, then $x \in \text{int}W^s(x)$.*

Proof. Let $y \in \text{int}W^s(x)$. There exists entourage α_1 such that $\alpha_1^2 \subset \alpha$, $\alpha_1[y] \subset W^s(x)$, where α is given in Theorem 2.3.

By the POTP of f , every β -pseudo orbit is α -traced for some entourage β . Since f is uniformly continuous, there exists entourage β_0 such that

$$(1) \quad (u, v) \in \beta_0 \text{ implies } (f(u), f(v)) \in \beta.$$

Choose entourage γ such that $\gamma^2 \subset \beta_0$. By $y \in W^s(x)$, there is $N \in \mathbb{N}$ such that $(f^n(x), f^n(y)) \in \gamma$ for all $n > N$.

By the continuity of f^N , there exists neighborhood U of x such that $f^N(x') \in \gamma[f^N(x)]$ for all $x' \in U$. We claim that $U \subset W^s(x)$. Let $x' \in U$. Since $(f^N(x'), f^N(x)) \in \gamma^{-1}$ and $(f^N(x), f^N(y)) \in \gamma$, we have $(f^N(x'), f^N(y)) \in \gamma^2 \subset \beta_0$.

By (1), the sequence $\{x_n\}$ defined by

$$x_n = \begin{cases} f^n(y) & \text{for } n \leq N \\ f^n(x) & \text{for } n > N, \end{cases}$$

is a β -pseudo orbit. Hence it is α_1 -traced by some point $y' \in X$. This implies in particular.

(2)

$$(y', y) \in \alpha,$$

(3)

$$(f^n(y'), f^n(x')) \in \alpha_1 \text{ for all } n > N.$$

(2) implies $y' \in \alpha_1[y] \subset W^s(x)$ and there is N' such that $(f^n(y'), f^n(x)) \in \alpha_1$ for all $n \geq N'$ which together with (3) imply $(f^n(x), f^n(x')) \in \alpha_1^2 \subset \alpha$ for all $n \geq M$, where $M = \max\{N, N'\}$. Thus $x' \in f^{-M}(W_\alpha^s(f^M(x))) \subset W^s(x)$. Hence the proof is completed. \square

PROPOSITION 3.2. *Let α be any entourage. If $x \in \text{int}W^s(x)$, then $x \in \text{int}W_\alpha^s(x)$.*

Proof. Let $x \in \text{int}W^s(x)$ and α be entourage. It is enough to have $N \in \mathbb{N}$ and a neighborhood U_1 of x such that $(f^n(x), f^n(y)) \in \alpha$ for all $n > N$ and all $y \in U_1$.

Since f^n are continuous for all $0 \leq n \leq N$, there exists neighborhood U_2 of x such that $(f^n(x), f^n(y)) \in \alpha$ for all $0 \leq n \leq N$ and all $y \in U_2$.

Then $U = U_1 \cap U_2$ is a neighborhood of x and $U \subset W_\alpha^s(x)$. Assume that is not true, that is,

(4) for any number $n \in \mathbb{N}$ and any neighborhood U of x there is entourage α and $y \in U$ such that $(f^n(x), f^n(y)) \notin \alpha$.

By $x \in \text{int}W^s(x)$, there exists entourage α_1 such that $\alpha_1^2 \subset \alpha$ and $\alpha_1[x] \subset W^s(x)$. From the POTP of f , there is entourage β such that every β -pseudo orbit is α_1 -traced.

Since f is uniformly continuous, there exists entourage γ such that $(u, v) \in \gamma$ implies $(f(u), f(v)) \in \beta$. \square

To complete the proof of Proposition 3.2, we need the following.

LEMMA 3.3. *There is a β -pseudo orbit (z_n) sequences $(l_i)_{i=1}^\infty, (n_i)_{i=2}^\infty$ in \mathbb{N} with $0 = l_1 < n_2 < l_2 < n_3 < \dots$ such that*

(5)

$$(f^{n_i}(x), z_{n_i}) \notin \alpha,$$

(6)

$$z_{l_i} = f^{l_i}(x) \text{ for all } i.$$

Proof. Define $z_n = f^n(x)$ for $n \leq 0$ and Put $l_1 = 0$. Assume that we have already defined numbers n_i, l_i for $i \leq j$ and points z_n for $n \leq l_j$ such that (5) and (6) hold and $(f^n(z_n), z_{n+1}) \in \beta$ for $n < l_j$.

Since f^{l_j} is continuous, there is a neighborhood $U \subset W^s(x)$ of x such that $(f^{l_i}(x), f^{l_i}(p)) \in \gamma$ for all $p \in U$. By (4), $(f^{n_{j+1}}(x), f^{n_{j+1}}(y)) \notin \alpha$ for some $n_{j+1} > l_i$ and for some $y \in U$. Because $y \in W^s(x)$, $(f^{l_{j+1}}(x), f^{l_{j+1}}(y)) \in \gamma$ for some $l_{j+1} > n_{j+1}$.

Put $z_n = f^n(y)$ for $l_j < n < l_{j+1}$ and $z_{l_{j+1}} = f^{l_{j+1}}(x)$. Since $(f^{l_j}(x), f^{l_j}(y)) \in \gamma$, we have $(f(f^{l_j}(x)), f(f^{l_j}(y))) = (f(z_{l_j}), f^{l_{j+1}}(y)) = (f(z_{l_j}), z_{l_{j+1}}) \in \beta$.

Also $(f^{l_{j+1}-1}(x), f^{l_{j+1}-1}(y)) \in \gamma$ implies $(f(f^{l_{j+1}-1}(y)), f(f^{l_{j+1}-1}(x))) = (f(z_{l_{j+1}-1}), f^{l_{j+1}}(x)) = (f(z_{l_{j+1}-1}), z_{l_{j+1}}) \in \beta$.

Therefore (5) and (6) are also satisfied for $j + 1$. Thus the Lemma is proved. \square

Now to finish the proof of Proposition 3.2, apply to Lemma 3.3. Take a point x' , α_1 -tracing the β -pseudo orbit (z_n) . Then we get $(f^n(x'), z_n) \in \alpha_1$ for all $n \in \mathbb{Z}$.

In Particular, $(x', z_0) = (x', x) \in \alpha_1$. By $x' \in \alpha_1[x] \subset W^s(x)$, we get $(f^n(x'), f^n(x)) \rightarrow 0$ as $n \rightarrow \infty$. Since $(f^{n_i}(x), z_{n_i}) \in \alpha_1$, we conclude that $(f^{n_i}(x), z_{n_i}) \subset \alpha_1^2 \subset \alpha$.

We have a contradiction to the fact that $(f^{n_i}(x), z_{n_i}) \notin \alpha$ by (5). This contradiction proves that $x \in \text{int}W_\alpha^s(x)$.

PROPOSITION 3.4. *Let α_0, α_1 be entourages with $\alpha_1^4 \subset \alpha$ on compact uniform space X . If $x \in \text{int}W_{\alpha_1}^s(x)$, then $L^+(x)$ is a periodic orbit which equal to $\{y, f(y), \dots, f^{k-1}(y)\}$ and $y \in \text{int}W^s(x)$ for all $y \in L^+(x)$.*

Proof. Let $x \in \text{int}W_{\alpha_1}^s(x)$. There exists entourage $\alpha_2 \subset \alpha_1$ such that $\alpha_2[x] \subset W_{\alpha_1}^s(x) \subset W^s(x)$.

By the compactness of X , it follows that $L^+(x) \neq \emptyset$. Also, it is known that $L^+(y) \subset L^+(x)$ for all $y \in L^+(x)$.

We claim that $L^+(y) = L^+(x)$ for all $y \in L^+(x)$. Let $z \in L^+(x)$. For each neighborhood U of z , it follows that $\beta_0[z] \subset U$ for some entourage $\beta_0 \subset \alpha_0$. Choose entourage β_1 such that $\beta_1^3 \subset \beta_0$. By $f \in \text{POTP}$, there is entourage γ such that every γ -pseudo orbit is β_1 -traced.

Also, $y \in L^+(x)$ implies $(f^N(x), y) \in \gamma$ for some $N \in \mathbb{N}$.

Let us defined by

$$z_n = \begin{cases} f^n(x) & \text{for } n < N \\ f^{n-N}(y) & \text{for } n \geq N. \end{cases}$$

Since a sequence (z_n) is a γ -pseudo orbit, (z_n) is β_1 -traced by some point $x' \in X$. Because $(x', z_0) = (x', x) \in \beta$, it follows that $x' \in \beta_1[x] \subset \beta_0[x] \subset \alpha_2[x] \subset W^s(x)$.

So we get the following, respectively,

$$\begin{aligned} (f^n(x), f^n(x')) &\in \beta \text{ for large } n, \\ (f^n(x), z) &\in \beta_1 \text{ for infinitely many } n, \text{ and} \\ (f^n(x'), f^{n-N}(y)) &\in \beta_1 \text{ for } n \geq N. \end{aligned}$$

Thus $(f^n(x), z) \in \beta_1^3 \subset \beta_0$, so $\beta^n(y) \in \beta_0[z] \subset U$ for infinitely many n . Therefore we conclude that $z \in L^+(y)$.

Next we shall show that for each $y \in L^+(x)$, $y \in \text{int}W^s(y)$. Let $y \in L^+(x)$. Since f satisfy POTP, there exists entourage β such that every β -pseudo orbit is α_2 -traced. Choose entourage γ such that $\gamma^2 \subset \beta$. Also $y \in L^+(x)$ implies $(f^N(x), y) \in \gamma$ for some $N \in \mathbb{N}$.

To show that $\gamma[y] \subset W^s(y)$, let $z \in \gamma[y]$. We define two sequence $(y_n), (z_n)$ by the following

$$y_n = \begin{cases} f^n(x) & \text{for } n < N \\ f^{n-N}(y) & \text{for } n \geq N \end{cases} \text{ and } z_n = \begin{cases} f^n(x) & \text{for } n < N \\ f^{n-N}(x) & \text{for } n \geq N. \end{cases}$$

Then this sequences are both β -pseudo orbits, and they are α_2 -traced by points $y', z' \in X$, respectively. By $(y', y_0) = (y', x) \in \alpha_2$ and $(z', z_0) = (z', x) \in \alpha_2$, we have $y', z' \in \alpha_2[x] \subset W_{\alpha_1}^s(x)$.

we obtain for $n > 0$, respectively, the following

$$\begin{aligned} (f^{n+N}(z'), z_{n+N}) &\in \alpha_2 \subset \alpha_1, \\ (f^{n+N}(x), f^{n+N}(z')) &\in \alpha_1, \\ (f^{n+N}(x), f^{n+N}(y')) &\in \alpha_1, \text{ and} \\ (f^{n+N}(y), y_{n+N}) &\in \alpha_2 \subset \alpha_1. \end{aligned}$$

So we have $(f^n(y), f^n(z)) = (y_{n+N}, z_{n+N}) \in \alpha_1^4 \subset \alpha$.

Thus $z \in W_\alpha^s(y) \subset W^s(y)$. Hence since $\gamma[y] \subset W^s(y)$, we obtain that $y \in \text{int}W^s(y)$.

By Proposition 3.1, any neighborhood of a limit point contains a periodic point. Let $p \in W^s(y)$ be a periodic point. We claim that $p = y$. For $p \neq y$ we would find another periodic point $q \in W^s(x), q \neq p$. Hence $W^s(p) = W^s(y) = W^s(q)$.

So $(f^n(p), f^n(q)) \rightarrow 0$ as $n \rightarrow \infty$, which is impossible. Thus we conclude that $L^+(x) = L^+(y) = \{y, f(y), \dots, f^{k-1}(y)\}$, where $k > 0$ is a periodic of y . □

PROPOSITION 3.5. *Let α be an entourage. $x \in \text{int}W_\alpha^s(x)$ if and only if $W_\alpha^s(x) \cap H = \{x\}$ for some neighborhood H of x .*

Proof. Assume that $x \in \text{int}W_\alpha^s(x)$. Take a neighborhood H of x such that $H \subset W_\alpha^s(x) \cap \beta[x]$, where β is chosen in terms of α as in Theorem 2.3. For $y \in W_\alpha^s(x) \cap H$, We have $y \in W_\alpha^s(x) \cap W_\alpha^u(x) = \{h(x, x)\} = \{x\}$.

Suppose that $W_\alpha^u(x) \cap H = \{x\}$ for some neighborhood H of x .

The continuity of the mapping $h : \beta \rightarrow X$ in Theorem 2.3 provides a neighborhood U of x such that $y \in U$ implies $h(x, y) \in H$.

We may assume that α is symmetric. Since $x \in W_\alpha^s(y)$, we have $y \in W_\alpha^s(x)$ and so $U \subset W_\alpha^s(x)$. Thus $x \in \text{int}W_\alpha^s(x)$. \square

PROPOSITION 3.6. *Let α, α_f be an entourages. Let $x \in \text{int}W_\alpha^s(x)$ be a periodic point with periodic k and let $\alpha \subset \alpha_f$ (see Lemma 2.1). Then there is a neighborhood U of x such that*

$$(7) \quad f^k(U) \subset U,$$

$$(8) \quad \bigcap_{n=0}^{\infty} f^{nk}(U) = \{x\}.$$

Proof. Let $x \in U_1 \subset \text{int}W_\alpha^s(x)$ for some open U_1 . For every $n \geq 0$, it follows that $f^{nk}(U_1) \subset \alpha[x]$.

Define $U = \bigcup_{n=0}^{\infty} f^{nk}(U_1)$. Then U is a neighborhood of x and (7) holds true. Furthermore, $U \subset \alpha[x]$. Let $x' \in \bigcap_{n=0}^{\infty} f^{nk}(U_1)$. Then in particular $x' \in U$ and by (7), $x' \in f^{nk}(U)$ for any $n < 0$. So for any $n \in \mathbb{Z}$, $x' \in f^{nk}\alpha[x]$ and hence for all n , $(f_{n_k}(x'), (f_{n_k}(x))) = (f_{n_k}(x'), x) \in \alpha \subset \alpha_{f_k}$. So $x' = x$, and (8) is proved. \square

The following theorem 3.7 summarizes and completes all previous results.

THEOREM 3.7. *Let α, α_0 be entourages and $y \in L^+(x)$ for $x \in X$. Then for a point $x \in X$, the following are equivalent.*

- (AS) $\text{int}W_\alpha^s(x) \neq \emptyset$,
- (BS) $x \in \text{int}W_\alpha^s(x)$,
- (CS) (x) is open,
- (DS) $x \in \text{int}W_\alpha^s(x)$,
- (ES) $\text{int}W_\alpha^s(x) \neq \emptyset$,
- (FS) $\text{int}W_\alpha^u(x) \cap H = \{x\}$ for some neighborhood H of x ,
- (GS) $\text{int}W_\alpha^u(y) = \{y\}$,
- (HS) $\text{int}W_\alpha^u(y) = \{y\}$,

(IS) $L^+(x) = \{y, f(y), \dots, f^{k-1}(y)\}$ is a periodic orbit.

There is a neighborhood U of y such that

(9) $f^k(U) \subset U$,

(10) $\bigcap_{n=0}^{\infty} f^{nk}(U) = \{x\}$.

Proof. (AS) \implies (BS) by Proposition 3.1.

To see that (BS) \implies (CS) consider a point $z \in W^s(x)$. Then $W^s(z) = W^s(x)$ contains x in its interior. By Proposition 3.1, $z \in \text{int}W^s(z) = \text{int}W^s(x)$ and hence $W^s_\alpha(x)$ is open. (CS) \implies (DS) Proposition 3.2.

(DS) \implies (ES) is trivial.

(ES) \implies (AS) by Theorem 2.3 (C).

(DS) \implies (FS) by Proposition 3.4.

Thus we have proved equivalences among conditions (AS) to (FS).

Now we shall prove (AS) \implies (IS). By Proposition 3.2, $x \in \text{int}W^s_{\alpha_0}(x)$. By Proposition 3.2, $y \in \text{int}W^s(y)$ and y is periodic. Once again by Proposition 3.2, $y \in \text{int}W^s_{\alpha_1}(y)$ for small α_1 . Proposition 3.5 completes the proof.

To see (IS) \implies (HS), take some $z \neq y$. Then for all n large enough, $z \notin f^{nk}(U)$, as (9) implies the sequence $f^{nk}(U)$ decrease. So $f^{-nk}(z) \notin U$ for such n and $z \notin W^u(y)$. Hence $W^u(y) = \{y\}$.

(HS) \implies (GS) by Theorem 2.3 (C).

We prove (GS) \implies (DS). $W^u_\alpha(y) = \{y\}$ implies $W^u_\beta(y) = \{y\}$, where $\beta^2 \subset \alpha$ and by Proposition 3.4, $y \in \text{int}W^s_\beta(y) = \{y\}$. As $y \in L^+(x)$, then there is a number N such that $f^N(x) \in \text{int}W^s_\beta(y)$.

The continuity of f^n , $0 \leq n \leq N$, provides a neighborhood U of x such that for $z \in U$, $(f^n(x), f^n(z)) \in \alpha$ for $0 \leq n \leq N$ and $f^N(z) \in W^s_\beta(y)$. It is clear that $U \subset W^s_\alpha(x)$. So (DS) is true. The proof of Theorem 3.7 complete. \square

Applying Theorem 3.7 to the inverse of f we immediately get.

THEOREM 3.8. *Let $x \in X$ and let α, α_0 be entourages with $\alpha \subset \alpha_0$, and $y \in L^-(x)$. Then the following are equivalent.*

(AU) $\text{int}W^u(x) \neq \emptyset$,

(BU) $x \in \text{int}W^u(x)$,

(CU) $W^u(x)$ is open,

(DU) $x \in \text{int}W^u_\alpha(x)$,

(EU) $\text{int}W^u_\alpha(x) \neq \emptyset$,

(FU) $\text{int}W^s_\alpha(x) \cap H = \{x\}$ for some neighborhood H of x ,

(GU) $\text{int}W^s_\alpha(y) = \{y\}$,

(HU) $\text{int}W^s(y) = \{y\}$,

(IU) $L^-(x) = \{y, f(y), \dots, f^{k-1}(y)\}$ is a periodic point,
 There is a neighborhood U of y such that
 $f^{-n}(U) \subset U$, and $\bigcap_{n=0}^{\infty} f^{-nk}(U) = \{y\}$.

Assume in the sequel that x is a non-isolated point in the space X and $y \in L^+(X)$. Let us note the situations described in Theorem 3.7 and Theorem 3.8 exclude each other. To see this, compare conditions (DS) and (FU) . There is just on other interesting situation.

It is enough to combine the negation of a condition from (AS) to (IS) with the negation of a condition from (AU) to (IU) to get the description of this situation.

Taking advantage of (FS) to (FU), for example, we get
 (Y)

any neighborhood of x contains points

$$x_S \in W_\alpha^s(x) - \{x\}, x_U \in W_\alpha^u(x) - \{x\}, x_S \neq x_U.$$

By Theorem 2.3, $x_S \neq x_U$ implies $W_\alpha^s(x) \cap W_\alpha^u(x) = \{x\}$.

LEMMA 3.9. *If a point x is periodic, then condition (Y) is equivalent to*

(X)

any neighborhood of x contains points

$$x_S \in W^s(x) - \{x\}, x_U \in W^u(x) - \{x\}, x_S \neq x_U.$$

Proof. (Y) \implies (X) by Theorem 2.3 (C).

To see (Y) \implies (X), take advantage of (HS) \iff (FS) and (HU) \iff (FU) in the case $y = x \in L^+(x) = L^-(x)$. \square

Now, it makes sense to call a periodic point a sink, if the situation described in Theorem 3.7 occurs, a source, if the situation described in Theorem 3.8 occurs, a saddle, if the situation described in Lemma 3.9 occurs.

Similarities to the hyperbolic case are clear. Thus we have

THEOREM 3.10. *Any periodic point is either a sink or a source or a saddle.*

4. Recurrent points

Let (X, \mathcal{U}) be an uniform space and $f : X \rightarrow X$ be a homeomorphism. A *fixed point* of f is a point x such that $f(x) = x$; a *periodic point* is a fixed point for an iterate of f . We denote by $Per(f)$ the set of periodic points.

The *negative and positive limit set* of f are defined by $L^-(f) = \bigcup_{x \in X} L^-(x)$ and $L^+(f) = \bigcup_{x \in X} L^+(x)$.

The *nonwandering set* is defined by $\Omega(f) = \{x \in X \mid \text{for all neighborhood } U \text{ of } x, \text{ there exists } n \neq 0 \text{ such that } f^n(U) \cap U \neq \emptyset\}$.

And the *chain recurrent set* is defined by $CR(f) = \{x \in X \mid \text{for all entourage } \alpha, \text{ there exists } \alpha\text{-chain from } x \text{ to itself}\}$, where an α -chain from x to itself is a finite α -pseudo orbit $\{x_0, \dots, x_n\}$, $x = x_0 = x_n$, $n > 0$.

Then by definition $\Omega(f)$ and $CR(f)$ are closed invariant sets, and $Per(f) \subset L^-(f) \cap L^+(f) \subset L^-(f) \cup L^+(f) \subset \Omega(f) \subset CR(f)$.

THEOREM 4.1. *Let (X, \mathcal{U}) be compact uniform space. Let f be an expansive homeomorphism for which f has POTP. Then*

$$\overline{Per(f)} = \overline{L^-(f)} = \overline{L^+(f)} = \Omega(f) = CR(f).$$

Proof. We must show that $CR(f) \subset \overline{Per(f)}$. Let $x \in CR(f)$ and entourage α_f for expansiveness of f . Fixed an entourage α such that $\alpha^2 \subset \alpha_f$

POTP provided a β -corresponding to this α , i.e., for any entourage α there is entourage β such that any β -pseudo orbit is α -traced. There is a β -chain $\{x_1, x_2, \dots, x_N\}$ from x to itself.

The sequence (x'_n) , $n \in \mathbb{Z}$ defined by $x'_n = x_i$ if $n = i \pmod{N}$ is a β -pseudo orbit.

It is α -traced by some point $y \in X$, but also by $f^N(y)$. By $(f^n(y), x'_n) \in \alpha$ and $(f^n(f^N(y)), x'_n) \in \alpha$ for $n \in \mathbb{Z}$, it follows that $(f^n(f^N(y)), f^n(y)) \in \alpha^2 \subset \alpha_f$.

Thus $f^N(y) = y$ and so $y \in Per(f)$. Since $(y, x'_0) = (y, x) \in \alpha$, we get $y \in \alpha[x]$. Consequently, we conclude that $x \in \overline{Per(f)}$. □

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